Unit-3

Solution of Non-Homogeneous LDE with Constant coefficient : ~~Consider~~.

Consider a NHLDEWC of order n as.

$$a_0 (y^n)_x + a_1 y^{n-1}(x) + \cdots + a_n(y)(x) = r(x)$$

where $a_0 \neq 0$, $a_1 \cdots a_n$ are constants.

$$\& \; r(x) \neq 0.$$

General solution of eq. (1) has two parts one is called as complementary function or solution (y_c) and 2nd part is particular integral or solution (y_p) then general solution of (1) is $y = y_c + y_p$

$$y = y_c + y_p = C.F + P.I$$

for finding complementary function:

Now, from (1) consider HLDEWC by taking $r(x) = 0$

$$a_0 y^n(x) + a_1 y^{n-1}(x) + \cdots + a_n y(x) = 0 \quad \text{(II)}$$

Find general solution of HLDEWC given in (II) and it is known as complementary function of eq. (1).

For finding Particular Integral: (P.I.)

Consider eq (1).

$$a_0 y^n(x) + a_1 y^{n-1}(x) \ldots + a_n y = r(x)$$

$$D = \frac{d}{dx}, \quad D^2 = \frac{d^2}{dx^2} \ldots$$

$$a_0 D^n y + a_1 D^{n-1} y + \ldots a_n y = r(x)$$

$$\left[a_0 D^n + a_1 D^{n-1} \ldots + a_n\right] y = r(x)$$

$$f(D) y = r(x)$$

where $f(D) = \{a_0 D^n + a_1 D^{n-1} + \ldots a_n\}$

$$y = \frac{1}{f(D)} r(x)$$

Case : 1 :

If $r(x) = e^{ax}$, then

$$\frac{1}{f(D)} r(x) = \frac{1}{f(D)} (e^{ax}) = \frac{1}{f(a)} e^{ax},$$

$$f(a) \neq 0.$$

$\Rightarrow$ If $F(a)=0$, [Case of failure] then,

$$\frac{1}{F(D)} e^{ax} = x \frac{1}{F'(a)} e^{ax}, \; F'(a) \neq 0$$

If $F'(a)=0$ [case of failure] then

$$\frac{1}{F(D)} (e^{ax}) = x^2 \frac{1}{F''(a)} e^{ax}, \; F''(a) \neq 0.$$

Case : 2 :

If $r(x) = \sin(ax+b)$ or $\cos(ax+b)$

$$\frac{1}{F(D^2)} \sin(ax+b) = \frac{1}{F(-a^2)} \sin(ax+b), \; F(-a^2) \neq$$

If $F(-a^2) = 0$, (case of failure)

$$\frac{1}{F(D^2)} \sin(ax+b) = \frac{x}{F'(-a^2)} \sin(ax+b),$$

$F'(-a^2) \neq 0$

Case : 3 :

If $r(x) = e^{ax} V(x)$

$$\frac{1}{F(D)} r(x) = \frac{1}{F(D)} [e^{ax} V(x)]$$

$$= e^{ax} \frac{1}{F(D+a)} V(x)$$

* Case 4:

Let $F(D) = D$

$$\therefore \frac{1}{F(D)} \phi(x) = \frac{1}{D} \phi(x) = \int \phi(x) dx$$

* Case 5:

[General formula]

$$\frac{1}{(D-a)} \phi(x) = \int e^{ax} \int e^{-ax} dx$$

$$\frac{1}{(D-a)} \phi(x) = e^{ax} \int e^{-ax} \phi(x) dx.$$

* Case 6:

If $\phi(x) = x^n$, n is +ve integer.

$$\frac{1}{F(D)} \phi(x) = \frac{1}{F(D)} x^n = [F(D)]^{-1} x^n$$

we expand $[F(D)]^{-1}$ using binomial theorem and then apply it to x^n.

Ques: Solve $(D^2+5D+4)y = 18e^{2x}$

Sol^n: $y'' + 5y' + 4y = 18e^{2x}$

A.E is $m^2+5m+4=0$

$(m+4)(m+1)=0$

$m=-1,-4$ are the roots of A.E

$\therefore$ Complementary function $y_c = C_1e^{-x} + C_2e^{-4x}$

Now, particular Integral $= \frac{1}{(D^2+5D+4)}[18e^{2x}]$

$= 18 \frac{1}{D^2+5D+4} e^{2x}$

$\left[\frac{1}{F(D)} e^{ax} = \frac{1}{F(a)} e^{ax} \quad F(a) \neq 0 \right.$

$= 18 \times \frac{1}{2^2+5\times 2+4} e^{2x}$

$= \frac{18}{18} \times e^{2x} = e^{2x}.$

Now, general solution of given eq is

$y = y_c + y_p = C_1e^{-x} + C_2e^{-4x} + e^{2x}$

Ques: $(D^2-6D+9)y = 14e^{3x}.$

Sol^n: m^2-6m+9

$m^2-3m-3m+9$

$m(m-3)-3(m-3)$ $\boxed{m=3}$

$(C_1 + C_2 x)e^{3x}$.

Now, PI =

$$14 * \frac{e^{3x}}{3^2 - 6\times 3 + 9} = 18 \frac{1}{18-18} e^{3x}.$$

Case of failure:

$$= 14x \frac{1}{2D-6} e^{3x} = 14x \frac{1}{2\times 3 - 6} e^{3x}$$

Again, case of failure.

$$= \frac{14x^2}{2} e^{3x} = 7x^2 e^{3x}$$

$\therefore$ General solution is $y = y_c + y_p$

$$(C_1 + C_2 x)e^{3x} + 7x^2 e^{3x}.$$

Ques: $(D^2 + 16)y = \cos 2x$

DF ~~wf~~ $A \cdot E = m^2 + 16 = 0$

$m = \pm 4i$

y_c = Complementary function =

$C_1 \cos 4x + C_2 \sin 4x$

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Now particular integral is $y_p = \frac{1}{(D^2+16)} \cos 2x$

$= \frac{1}{(-4+16)} \cos 2x = \frac{\cos 2x}{12}$

$\left\{ \begin{array}{l} \frac{1}{F(D^2)} \cos(ax+b) \\ = \frac{1}{F(-a^2)} \cos(ax+b) \\ a=2 \\ a^2=4 \\ -a^2=-4. \end{array} \right.$

G.S is $y = y_c + y_p$

$C_1 \cos 4x + C_2 \sin 4x + \frac{\cos 2x}{12}$

ques: $(2D^2 - 5D + 3)y = \sin x.$

sol^n. $2D^2$. A.E is $2m^2 - 5m + 3$

$2m^2 - 3m - 2m + 3$

$m(2m-3) - 1(2m-3)$

$(m-1)\ (2m-3)$

$m=1 \quad m = \frac{3}{2}$

$y_c = C.F = C_1 e^{x} + C_2 e^{\frac{3}{2}x}$

Now, $y_p = P.I = \frac{1}{(2D^2 - 5D + 3)} \sin x$

$\left\{ \begin{array}{l} a=1 \\ a^2=1 \\ -a^2=-1 \end{array} \right.$

$= \frac{1}{2(-1) - 5D + 3} \sin x.$

$= \frac{1}{-5D+1} \sin x.$

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$$= (5D+1)\left[\frac{1}{(1+5D)(1-5D)}\right]\sin x.$$

$$\frac{5D+1}{1-25D^2}\sin x.$$

$$= (5D+1)\left[\frac{1}{1-25D^2}\sin x\right] = (5D+1)\left[\frac{1}{1-25(-1)}\sin x\right]$$

$$= \frac{1}{26}(5D+1)\sin x = \frac{1}{26}[5D\sin x + \sin x]$$

$$= \frac{1}{26}\left[\frac{5d}{dx}\sin x + \sin x\right] = \frac{1}{26}[5\cos x + \sin x]$$

G.S. is $y_c + y_p = C_1 e^{x} + C_2 e^{3/2 x} + \frac{1}{26}[5\cos x + \sin x]$

Ans: A.E is $m^2+3=0 \Rightarrow m = \pm\sqrt{3}i$

$$C.F = y_c = C_1\cos\sqrt{3}x + C_2\sin\sqrt{3}x.$$

Now, $y_p = P.I = \frac{1}{D^2+3}\cos(\sqrt{3}x)$

$$= \frac{1}{-3+3}\cos\sqrt{3}x)$$

Case of failure

$$\frac{1}{D}(\phi(x))$$

$$= \frac{x \times 1}{2D}\cos(\sqrt{3}x) = \frac{x}{2}\int \cos\sqrt{3}x\,dx \quad = \int \phi(x)\,dx$$

$$= \frac{x}{2}\frac{\sin\sqrt{3}x}{\sqrt{3}} = \frac{x}{2\sqrt{3}}\sin\sqrt{3}x$$

or

$$\frac{x}{2}\frac{D}{D^2}\cos\sqrt{3}x$$

$$= \frac{x}{2}\frac{D}{-3}\cos\sqrt{3}x$$

$$= -\frac{x}{6}\frac{d}{dx}[\cos\sqrt{3}x]$$

$$= -\frac{x}{6}(-\sin\sqrt{3}x)\sqrt{3}$$

$$= \frac{x\sin\sqrt{3}x}{2\sqrt{3}}$$

$\therefore$ General solution is $y = y_c + y_p$

$$= C_1\cos\sqrt{3}x + C_2\sin\sqrt{3}x + \frac{x}{2\sqrt{3}}\sin\sqrt{3}x$$

Ques: $(D^2 - 4D + 5)y = 24e^{2x}\sin x.$

Sol^n. A.E is $m^2 - 4m + 5 = 0$

$$m = \frac{-(-4) \pm \sqrt{(-4)^2 - 4\times1\times5}}{2\times1}$$

$$m = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$$

$2+i,\ 2-i$ are the roots.

$$C.F = y_c = e^{2x}[C_1\cos x + C_2\sin x]$$

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$$y_p = P.I = \frac{1}{(D^2-4D+5)} 24\, e^{2x} \sin x.$$

$$= 24 \frac{1}{(D^2-4D+5)} e^{2x} \sin x.$$

$$= 24 \times e^{2x} \frac{1}{(D+2)^2-4(D+2)+5} \times \sin x.$$

$$= 24 \times e^{2x} \left(\frac{\sin x}{D^2+1} \right)$$

$$\left\{ \begin{array}{l} * \frac{1}{F(D)} e^{ax} V(x) \\ e^{ax} \frac{1}{F(D+a)} V(x) \end{array} \right.$$

$$= 24 \times e^{2x} \frac{\sin x}{-1+1}$$

$$\left\{ \begin{array}{l} a = 1 \\ a^2 = 1 \end{array} \right. \quad -a^2 = -1.$$

Case of failure.

$$= 24 e^{2x}\, x \frac{1}{2D} \sin x = \frac{24 x e^{2x}}{2} \int \sin x\, dx$$

$$= -12\, x e^{2x} \cos x$$

$$\therefore G.S = y = y_c + y_p$$

$$= e^{2x}[c_1 \cos x + c_2 \sin x] - 12 x e^{2x} \cos x.$$

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* Case - VII :

$$\frac{1}{F(D)}[x\,V(x)] = x\cdot\frac{1}{F(D)}V(x) + \frac{d}{dD}\left[\frac{1}{F(D)}\right]V(x)$$

Ques: Solve : $(D^2+3D+2)y = xe^x \sin x$

Sol^n: A.E is $m^2+3m+2=0 \Rightarrow (m+2)(m+1)=0$
$m=-2,-1$ are the roots.

Now, $P.I = Y_P = \frac{1}{(D^2+3D+2)}\{xe^x \sin x\}$

$$= e^x \frac{1}{(D+1)^2+3(D+1)+2}\{x\sin x\}$$

$$= e^x\left(\frac{1}{D^2+2D+1+3D+3+2}\right)x\sin x.$$

$$= e^x\left[x\times\frac{1}{D^2+5D+6}\sin x + \frac{d}{dD}\left[\frac{1}{D^2+5D+6}\right]\sin x\right]$$

$$= e^x\left[x\times\frac{1}{-1+5D+6}\sin x + (-1)(D^2+5D+6)^{-2}(2D+5)\sin x\right]$$

$$= e^x\left[x\times\frac{1}{5D+5}\sin x - \frac{2D+5}{(D^2+5D+6)^2}\sin x\right]$$

$$= e^{x}\left[\frac{x}{5}\times\frac{1}{(D+1)(D-1)}(D-1)\sin x - \frac{2D+5}{(-1+5D+6)^2}\sin x\right.$$

$$= e^{x}\left[\frac{x}{5}\left(\frac{D-1}{-1-1}\right)\sin x + \frac{2D+5}{25[\times 2]D}\sin x\right.$$

$$= e^{x}\left[\frac{-x}{10}\left[\frac{d}{dx}\sin x - \sin x\right] - \frac{2D+5}{(25\times 2)D}\sin x\right]$$

$$= e^{x}\left[\frac{-x}{10}\cos[x - \sin x] - \frac{2D+5}{5D}\int \sin x\,dx\right]$$

$$= e^{x}\left[\qquad \text{''} \qquad + \frac{(2D+5)}{50}\cos x\right]$$

$$= e^{x}\left[\qquad \text{''} \qquad + \frac{1}{50}\left\{-2\sin x + 5\cos x\right\}\right]$$

$$y_p = e^{x}\left[\frac{-x}{10}(\cos x - \sin x) + \frac{1}{50}(-2\sin + 5\cos x)\right]$$

General solution is $y = y_c + y_p$

Ques: Solve $(D^2+6D+9)y = 4x^2-1$

Sol: A.E is $m^2+6m+9=0 \Rightarrow (m+3)^2=0$

$\Rightarrow m = -3, -3$ are the roots.

$\therefore y_c = C.F = (c_1+c_2x)e^{-3x}$

Now, $P.I = y_p = \dfrac{1}{(D^2+6D+9)}(4x^2-1)$

$$= \frac{1}{9\left[1+\frac{D^2+6D}{9}\right]}[4x^2-1]$$

$$= \frac{1}{9}\left[1+\frac{D^2+6D}{9}\right]^{-1}(4x^2-1)$$

$\cancel{(1+x)^n = 1+nx+n(n-1)x^2 + n(n-1)(n-2)}$

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots$$

[n is not a positive integer, $|x|<1$.

$$\frac{1}{9}\left[1+(-1)\left(\frac{D^2+6D}{9}\right)+\frac{(-1)(-1-1)}{2!}\left(\frac{D^2+6D}{9}\right)^2\right.$$

$$\left.+\frac{(-1)(-1-1)(-1-2)}{3!}\left(\frac{D^2+6D}{9}\right)^3+\cdots\infty\right](4x^2-1)$$

$$= \frac{1}{9}\left[1 - \left(\frac{D^2+6D}{9}\right) + \frac{D^4+36D^2+12D^3}{81} - \frac{D^6+216D^3+18D^5+108D^4}{9^3} + \cdots \infty\right](4x^2-1)$$

$$= \frac{1}{9}\left[(4x^2-1) - \frac{1}{9}\left[8+6(8x)\right] + \frac{1}{81}\left[0+36(8)+0\right] - \frac{1}{729}\cdot(0.) \cdots \infty\right]$$

$$= \frac{1}{9}\left[4x^2-1) - \frac{8+48x}{9} + \frac{4\times 8}{9}\right]$$

$$= \frac{1}{9}\left[4x^2-1) - \frac{8+48}{9} + \frac{32}{9}\right.$$

G.S. $y = y_c + y_p$

Ques: Solve $(D^2+3D+2)y = e^{e^x}$

Sol^n. A.E is $m^2+3m+2=0$

$(m+2)(m-1)=0$

$m=-1,-2$ are the roots.

$\therefore y_c = c_1 e^{-x} + c_2 e^{-2x}$.

Now $y_p = \dfrac{1}{(D^2+3D+2)} e^{e^x}$

$$y_p = \frac{1}{(D+1)(D+2)} e^{e^x}$$

$$= \left[\frac{1}{D+1} - \frac{1}{D+2}\right] e^{e^x}.$$

$$y_p = \frac{1}{(D+1)(D+2)} e^{e^x}$$

$$= \frac{1}{(D+1)} e^{e^x} - \frac{1}{(D+2)} e^{e^x}$$

$$= \frac{1}{D-(-1)} e^{e^x} - \frac{1}{D-(-2)} e^{e^x}$$

$$\frac{1}{(D+1)(D+2)} = \frac{A}{D+1} + \frac{B}{D+2}$$

$1 = A(D+2) + B(D+1)$

$= A+B=0 \Rightarrow A=-B$

$2A+B=1$

$-2B+B=1$

$\boxed{B=-1}$

$\boxed{A=1}$

$$= \int e^{e^x} e^x dx - e^{-2x} \int e^{e^x} \cdot e^{2x} dx$$

$$y_p = e^{-x} I_1 - e^{-2x} I_2 \quad \text{— (1)}$$

Now, $I_1 = \int e^{e^x} dx$, put $e^x = t \Rightarrow e^x dx = dt$

$$= \int e^t dt = e^t = e^{e^x}$$

Now $I_2 = \int e^{e^x} \cdot e^{2x} dx = \int e^{e^x} e^x \cdot e^x dx$

$$= I_2 = \int e^t \cdot t\, dt = te^t - e^t = (t-1)e^t$$

$$= (e^x - 1) e^{e^x}$$

$$y_p = e^{-x} e^{e^x} + e^{-2x}(e^x - 1) e^{e^x}$$

$$= e^{-x} e^{e^x} - e^{-x} e^{e^x} + e^{-2x} e^{e^x}$$

$$= \cancel{e^{-x} e^{e^x}} - \cancel{e^{-x} e^{e^x}} + e^{-2x} e^{e^x}$$

$$= e^{-2x} e^{e^x}$$

G.S is $y = y_c + y_p$

⇒ Method of variation of parameters: Consider a NH LDEWC of order two as $\frac{d^2y}{}$

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy(x) = r(x),$$ where P & Q are constants & $r(x) \neq 0$.

Let $y_c = C.F = C_1 y_1 + C_2 y_2$.

find $W_{y_1, y_2} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

Hence $y_1 = -y_1 \int \frac{y_2 r(x)}{W} dx + y_2 \int \frac{y_1 r(x)}{W} dx$.

Ques: solve using MVOP, $y'' + y = \tan x$

Sol^n: $\frac{d^2y}{dx^2} + y = \tan x$,

A.E = is $m^2 + 1 = 0 \Rightarrow m = \pm i$ are the roots.

$\therefore y_c = C_1\cos x + C_2 \sin x$,

$\therefore y_1 = \cos x$ & $y_2 = \sin x$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$= \cos^2 x + \sin^2 x = 1$

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Now, $y_p = -y_1 \int \frac{y_2\, r(x)}{W} dx + y_2 \int \frac{y_1\, r(x)}{W} dx$

$= -\cos x \int \frac{\sin x \tan x}{1} + \sin x \int \frac{\cos x \tan x}{1} dx$

$= -\cos x \int \frac{1-\cos^2 x}{\cos x} dx + \sin x \int \cos x \frac{\sin x}{\cos x} dx$

$= -\cos x \left[\log|\sec x + \tan x| - \sin x \right] + \sin x (-\cos x)$

$= -\cos x \log|\sec x + \tan x| + \cancel{\sin x \cos x} - \cancel{\sin x \cos x}$

$G.S = y = y_c + y_p = c_1 \cos x + c_2 \sin x - \cos x \log|\sec x + \tan x|.$

Ques: $y'' + y = \sec x$

Sol^n: A.E is $m^2 + 1 = 0$ $\quad m = \pm i$

$y_c = c_1 \cos x + c_2 \sin x.$

Here $y_1 = \cos x$ & $y_2 = \sin x \Rightarrow W(y_1, y_2) = 1$

Now, $y_p = -y_1 \int \frac{y_2\, r(x)}{W} dx + y_2 \int \frac{y_1\, r(x)}{W} dx$

$- \cos x \int (\sin x - \sec x) dx + \sin x \int \cos x - \sec x \, dx$

$= -\cos x \left[-\log|\cos x| \right] + \sin x [x]$

$= \cos x \log|\cos x| + x\sin x.$

$G.S = y = y_c + y_p.$

Ques: Solve using MVOP, if two linearly independent solution are given to us.

$$x^2y'' + xy' - y = x^3, \quad y_1 = x, \; y_2 = \frac{1}{x}$$

Sol^n. $\therefore W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & 1/x \\ 1 & -1/x^2 \end{vmatrix}$

$= -1/x - 1/x = \frac{-2}{x}.$

Now, $y_p = -y_1\int \frac{y_2 r(x)}{W} dx + y_2\int \frac{y_1 r(x)}{W} dx$

$= x\int \frac{x}{x(-2/x)} dx + \frac{1}{x}\int \frac{x \cdot x}{(-2/x)} dx$

for $r(x)$ divide the eq. with $r(x)$

$r(x) = \frac{x^3}{x^2} = x.$

$= \frac{-x}{-2}\left[\frac{x^2}{2}\right] + \frac{1}{-2x}\left[\frac{x^4}{4}\right]$

$= \frac{+x^3}{4} - \frac{x^3}{8} = \frac{x^3}{8}.$

$G.S = y = y_c + y_p = c_1y_1 + c_2y_2 + \frac{x^3}{8}$

$= c_1 x + \frac{c_2}{x} + \frac{x^3}{8}.$

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* Method of undetermined coefficient:

Consider a NHLDEWC of order n, as

$$a_0 y^n + a_1 y^{n-1} + a_2 y^{n-2} + \cdots + a_n y = r(x)$$

$a_0 \neq 0$, $a_1, a_2 \cdots a_n$ are all constants &
$r(x) \neq 0$.

Find y_c [complementary function], now in this method we assume y_p [particular integral] as discussed below.

(i) If $r(x) = e^{ax}$, then we assumed y_p as $c_1 e^{ax}$ where c_1 is underdetermined coefficient.

(ii) If $r(x) = \cos mx$ or $\sin mx$, then y_p will be assumed as $c_1 \cos mx + c_2 \sin mx$, where c_1, c_2 are undertermined coefficient.

(iii) If $r(x) = x^m$, where m is the +ve integral then y_p is assumed as $a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$, where $a_0, a_1 \cdots a_m$ are undertermined coefficient.

(iv) If $r(x) = e^{ax} \sin bx$ or $e^{ax} \cos bx$, then y_p will be assumed as ~~$y_p = e^{ax}$~~

$y_p = e^{ax}[c_1 \cos bx + c_2 \sin bx]$ where $c_1 c_2$ are undetermined coefficient

Note

While assuming y_p, it should be totally different from y_c. If in case a term is common in y_c & y_p, then multiply that term with x or x^m {depending upon situation}.

Ques: Solve using method of undetermined coefficient.

$$y'' - 3y' - 10y = \underbrace{1+x^2} \rightarrow (1).$$

Sol^n: A.E of eq $m^2 - 3m - 10 = 0$

$\Rightarrow (m-5)(m+2) = 0$

$m = -2, 5$ are roots.

$\therefore y_c = C.F = C_1 e^{-2x} + C_2 e^{5x}$

Now, we assume $y_p = a_0 + a_1 x + a_2 x^2$, where a_0, a_1, a_2 are undetermined coefficient. $\rightarrow$ put in (1)

$$\frac{d^2}{dx^2}\left[a_0 + a_1 x + a_2 x^2\right] - 3\frac{d}{dx}\{a_0 + a_1 x + a_2 x^2\}$$

$$-10\{a_0 + a_1 x + a_2 x^2\} = 1+x^2$$

$$2a_2 - 3\{a_1 + 2a_2 x\} - 10(a_0 + a_1 x + a_2 x^2) = 1+x^2$$

Equating coefficient of similar terms.

$x^2 \Rightarrow -10a_2 = 1$ $\boxed{a_2 = -1/10}$

$x \Rightarrow -6a_2 - 10a_1 = 0 \Rightarrow \boxed{a_1 = \frac{3}{50}}$

Constant $\Rightarrow 2a_2 - 3a_1 - 10a_0 = 1$

$\boxed{a_0 = \frac{-69}{500}}$

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$\therefore y_p = \frac{-69}{500} + \frac{3}{50}x - \frac{1}{10}x^2$

G.S. is $y = y_c + y_p = C_1 e^{-x} + C_2 e^{5x} +$

$\frac{-69}{500} + \frac{3}{50}x - \frac{1}{10}x^2$

<u>Ques</u>: $4y'' - y = e^{x} + e^{3x}$ — (1)

$\Rightarrow$ A.E. is $4m^2 - 1 = 0 \Rightarrow m = \pm\frac{1}{2}$.

$C.F. = y_c = C_1 e^{1/2 x} + C_2 e^{-1/2 x}$.

Now, let us assume y_p as $y_p = a_1 e^{x} + a_2 e^{3x}$,

put in (1). where a_1, a_2 are underdined coefficient.

$4[a_1 e^{x} + a_2 e^{3x}] - [a_1 e^{x} + a_2 e^{3x}] = e^{x} + e^{3x}$

Compare coefficient of similar term.

$\boxed{a_1 = 1/3}$ $\boxed{a_2 = \frac{1}{35}}$

$\therefore y_p = \frac{1}{3}e^{x} + \frac{1}{35}e^{3x}$

G.S. $= y = y_c + y_p = C_1 e^{x/2} + C_2 e^{-x/2} + \frac{e^{x}}{3} + \frac{e^{3x}}{35}$

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Ques: $y'' + 3y' + 2y = \underbrace{\cos x + \sin x}$ → (1)

Sol^n: A.E is $m^2 + 3m + 2 = 0 \Rightarrow m = -1, -2$ are the roots.

$C.F = y_c = C_1 e^{-x} + C_2 e^{-2x}$.

Now, Let us assume $y_p = \underbrace{b_1 \cos x + b_2 \sin x}$

↳ put in (1)

$\left[-b_1 \cos x - b_2 \sin x\right] + 3\left[-b_1 \sin x + b_2 \cos x\right] + 2\left[b_1 \cos x + b_2 \sin x\right] = \cos x + \sin x$

Compare Coefficient of similar terms.

$\cos x \Rightarrow -b_1 + 3b_2 + 2b_1 = 1 \Rightarrow b_1 + 3b_2 = 1$ — (ii)

$\sin x \Rightarrow -b_2 - 3b_1 + 2b_2 = 1 \Rightarrow -3b_1 + b_2 = 1$ — (iii)

& $\boxed{b_2 = \frac{2}{5}}$ $\boxed{b_1 = \frac{-1}{5}}$

$\therefore y_p = -\frac{1}{5} \cos x + \frac{2}{5} \sin x$

$G.S = y = y_c + y_p = C_1 e^{-x} + C_2 e^{-2x} - \frac{1}{5} \cos x + \frac{2}{5} \sin x.$

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Ques: $y'' + 2y' + 10y = e^{-x}\sin 3x$

Soln. A.E is $m^2 + 2m + 10 = 0$

$$m = \frac{-2 \pm \sqrt{4-40}}{2} = \frac{-2 \pm 6i}{2}$$

$$m = -1+3i, -1-3i$$

$$y_c = e^{-x}[c_1 \cos 3x + c_2 \sin 3x]$$

Now, let us assume $y_p = xe^{-x}[a_1 \cos 3x + a_2 \sin 3x]$

put in (1) to find a_1 & a_2

Ques: $y'' + 6y' + 9y = 26e^{-3x} + 5e^{2x}$

Sol^n: A.E $y'' + 6y' + 9y = 26e^{-3x} + 5e^{2x}$

$m^2 + 6m + 9 = 0 \Rightarrow (m+3)^2 = 0 \quad m = -3, -3$

$y_c = (C_1 + C_2 x)e^{-3x}$

Let us assume the $y_p = P.I$

$= a_1 e^{-3x} + a_2 e^{2x}$

put in (1) and

Complete at your own.

Ques: $y^{iv} - 3y'' - 4y = 60e^{2x}$

Sol^n. A.E: $m^4 - 3m^2 - 4 = 0 \quad (m^2 = t)$

$t^2 - 3t - 4 = 0 \Rightarrow (t-4)(t+1) = 0$

$m^2 = 4, -1 \Rightarrow m = \pm 2, \pm 1$

$y_c = C_1 e^{2x} + C_2 e^{-2x} + C_3 \cos x + C_4 \sin x$

Now, let $y.p = a_1 e^{2x} x$ put in (1)

{complete at your own}

Ques: $y'' - 6y' + 13y = 6e^{3x}\sin x \cos x.$

Sol^n. A.E $\quad m^2 - 6m + 3 = 0$

$$m = \frac{6 \pm \sqrt{36-72}}{2} = 3+2i,\ 3-2i.$$

$$y_c = e^{3x}\left[c_1 \cos 2x + c_2 \sin 2x\right]$$

Let $\quad y_p = e^{3x}\{a_1 \cos 2x + a_2 \sin 2x\} x$

Put in (1) | $x e^{3x} \cos 2x,$

Complete at your own. | $x e^{3x} \sin 2x.$

$$\frac{d}{dx}[uvw] = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$$

Euler Cauchy equation: Consider a n^{th} order euler-Cauchy eq. is as.

$$a_0 x^n y^n(x) + a_1 x^{n-1} y^{n-1}(x) + \dots a_n(y)(x) = r(x)$$

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} \dots +$$

$$a_n y(x) = r(x)$$

$r(x) = 0$ → HLDEWV

$r(x) \neq 0$ → NHLDEWV.

Ques: $x^2 \frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + 5y = \sin x$, → (NHLDEWV)

↳ (Euler-Cauchy)

Take $x = e^t$

$$a_0 x^n D^n y + a_1 x^{n-1} D^{n-1} y + a_2 x^{n-2} D^{n-2} y + \dots + a_{n-1} x dy + a_n y = r(x)$$

$$[a_0 x^n D^n + a_1 x^{n-1} D^{n-1} + a_2 x^{n-2} D^{n-1} \dots a_{n-1} xD + a_n] y = r(x)$$

Take $x = e^t$

$$xD = x\frac{d}{dx} = \theta, \quad x^2 D^2 = x^2 \frac{d^2}{dx^2} = \theta(\theta - 1)$$

$$x^3 D^3 = x^3 \frac{d^3}{dx^3} = \theta(\theta-1)(\theta-2) \dots$$

$$[f(\theta)]y = R(t) \rightarrow [\text{HLDEWC}] \text{ or NHLDEWC}$$

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Solve $x^2y'' + xy' - 4y = 0$

Sol^n $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = 0$ {HLDEWC}

$\Downarrow$

{Eulear Cauchy}

$\Rightarrow x^2D^2y + xDy - 4y = 0$

$\{x^2D^2 + xD - 4\}y = 0$

put $x = e^t$, $xD = \theta$, $x^2D^2 = \theta(\theta - 1)$

where $D = \frac{d}{dx}$ so $\theta = \frac{d}{dt}$, $\theta^2 = \frac{d^2}{dt^2}$

$[\theta(\theta-1) + \theta - 4]y = 0$

$(\theta^2 - \theta + \theta - 4)y = 0 \Rightarrow \theta^2y - 4y = 0$

$\Rightarrow \frac{d^2y}{dt^2} - 4y = 0$

A E is $m^2 - 4 = 0 \Rightarrow m = 2, -2$ are roots

$\therefore$ G.S $= y = c_1e^{2t} + c_2e^{-2t}$

$= y = c_1(e^t)^2 + c_2(e^t)^{-2} = c_1x^2 + c_2x^{-2}$

$y = c_1x^2 + \frac{c_2}{x^2}$

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Que: $x^2y'' - 2y = 2x + 6$

$$x^2 \frac{d^2y}{dx^2} - 2y = 2x+6$$

$$x^2 D^2y - 2y = 2x+6.$$

$$[x^2 D^2 - 2]y = 2x+6.$$

Take $x = e^t \Rightarrow xD = \theta,\ x^2D^2 = \theta(\theta-1)$

$$D = \frac{d}{dx},\quad \theta = \frac{d}{dt}$$

$$\Rightarrow [\theta(\theta-1) - 2]y = 2e^t + 6$$

$$[\theta^2 - \theta - 2]y = 2e^t + 6.$$

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 2e^t + 6 \Rightarrow \text{NHLDEwC.}$$

A.E is $m^2 - m - 2 = 0 \Rightarrow (m-2)(m+1) = 0$

$$m = 2, -1.$$

$$y_c = C.F = C_1e^{2t} + C_2e^{-t}$$

Now, $y_p = \frac{1}{(\theta^2-\theta-2)}[2e^t + 6]$

$$= 2 \times \frac{1}{(\theta^2-\theta-2)} e^t + 6 \times \frac{1}{\theta^2-\theta-2} \quad (1 = e^{(0)t})$$

$$= 2 \times \frac{1}{(1-1-2)} e^t + 6\frac{(1)}{(-2)} = -e^t - 3$$

Now,

$$G.S = y = y_c + y_p = C_1e^{2t} + C_2e^{-t} - e^t - 3$$

$$= C_1x^2 + \frac{C_2}{x} - x - 3$$

Ques: Solve; $x^2y'' - 3xy' + 3y = 2 + 3\log x$

Sol^n; $x^2\frac{d^2y}{dx^2} - 3x^2\frac{dy}{dx} + 3y = 2 + 3\log x$

[NHLDEwV]

[Euler - Cauchy]

$\Rightarrow x^2D^2y - 3xDy + 3y = 2 + 3\log x$

$\underbrace{[x^2D^2 - 3xD + 3]}y = 2 + 3\log x$

Take substitution

Take $x = e^t$

$xD = \theta$, $x^2D^2 = \theta(\theta - 1)$, where

$\theta = \frac{d}{dt}$, $D = \frac{d}{dx}$

$\Rightarrow [\theta(\theta-1) - 3\theta + 3] = 2 + 3$

$[\theta^2 - 4\theta + 3]dy = 2 + 3t$

$\Rightarrow \frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y = 2 + 3t$ [NHLDEwC]

A.E is $m^2 - 4m + 3 = 0$

$(m-1)(m-3) \Rightarrow m = 1, 3$ are the roots.

$y_c = C_1e^t + C_2e^{3t}$

Now, $y_p = \dfrac{1}{(\theta^2-4\theta+3)}(2+3t)$

$= \dfrac{1}{3}\left[\dfrac{1}{\left(1+\frac{(\theta^2-4\theta)}{3}\right)}(2+3t)\right].$

$\dfrac{1}{3}\left[1+\left(\dfrac{\theta^2-4\theta}{2}\right)\right]^{-1}(2+3t)$

$(1+x)^n = 1+nx+\dfrac{n(n-1)x^2}{2!}+\dfrac{n(n-1)(n-2)}{3!}+\cdots\infty$

$\dfrac{1}{3}\left[1+(-1)\left[\dfrac{\theta^2-4\theta}{3}\right]+\dfrac{(-1)(-1-1)}{2!}\left[\dfrac{\theta^2-4\theta}{3}\right]^2\right.$

$\left.+\cdots\cdots\infty\right](2+3t)$

$= \dfrac{1}{3}\left[(2+3t)-\dfrac{1}{3}\left[0-4(3)\right]+0+0+\cdots\infty\right]$

$= \dfrac{1}{3}\left[2+3t+4\right] = \dfrac{6+3t}{3} = t+2$

G.S $y = y_c+y_p = c_1e^t+c_2e^{3t}+t+2$

$\boxed{c_1x+c_2x^3+\log x+2}$

$x = e^t$
$\log x = t$

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Ques: Solve $x^2y'' + 2xy' = \cos(\log x)$

$$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = \cos(\log x)$$

{NHLDEWVC}
[Euler-Cauchy]

$\Rightarrow x^2D^2y + 2xDy = \cos(\log)$

$\Rightarrow [x^2D^2 + 2xD]y = \cos\log(x)$

Take $x = e^t$

$xD = \theta$, $x^2D^2 = \theta(\theta-1)$

where $D = \frac{d}{dx}$, $\theta = \frac{d}{dt}$.

$[\theta(\theta-1) + 2\theta]y = \cos(t)$

$[\theta^2+\theta]y = \cos t = \frac{d^2y}{dt^2} + \frac{dy}{dt} = \cos t$

A.E is $m^2+m=0 \Rightarrow m(m+1)=0 \Rightarrow m=0,-1$

$\therefore y_c = C_1e^{(0)t} + C_2e^{-t} = C_1 + C_2e^{-t}$

Now, $y_p = \frac{1}{\theta^2+\theta}(\cos t) =$

$y_p = \frac{1}{\theta \cancel{+1}}(\cos t)$

| $a = 1$
| $a^2 = 1$
| $-a^2 = -1$

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$$y_p = \frac{\theta+1}{\theta^2-1}\cos t$$

$$y_p = \frac{(\theta+1)(\cos t)}{\theta-1-1}$$

$$= \frac{1}{-2}\left[(\theta+1)\cos t\right]$$

$$= -\frac{1}{2}\left[\theta\cos t + \cancel{\theta t}^{\cos t}\right]$$

$$= \frac{1}{2}\sin t + \cos t$$

$$\therefore G.S = y = y_c + y_p = c_1 + c_2 e^{-t} + \frac{1}{2}\left[\sin t - \cos t\right]$$

$$= c_1 + \frac{c_2}{x} + \frac{1}{2}\left[\sin(\log x) - \cos(\log x)\right]$$

Ques: Solve: $2x^2 y'' + 3xy' - y = x,\ y(1) = 1,\ y(4) = \frac{41}{16}$

Soln. $2x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} - y = x \Rightarrow 2x^2 D^2 y + 3xDy - y = x$

$$[2x^2D^2 + 3xD - 1]y = x$$

Take $x = e^t$

$$\Rightarrow xD = \theta,\ x^2D^2 = \theta(\theta-1)$$

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$\Rightarrow [2\theta(\theta-1)+3\theta-1]y = e^t$

A·E is $2m^2+m-1=0 \Rightarrow 2m^2+2m-m-1$
$\Rightarrow (2m-1)(m+1)=0$
$\Rightarrow m = \frac{1}{2}, -1$

$\Rightarrow y_c = c_1 e^{\frac{1}{2}t} + c_2 e^{-t}$

Now $y_p = \frac{1}{2\theta^2+\theta-1} e^t = \frac{1}{2} e^t$

$\therefore$ G·S $= y = y_c + y_p = c_1 e^{t/2} + c_2 e^{-t} + \frac{1}{2}e^t$

$\Rightarrow y = c_1\sqrt{x} + \frac{c_2}{x} + \frac{x}{2}$ — (1)

$\Rightarrow y(1) = 1.$

$\Rightarrow y = 1$ when $x = 1.$

$\boxed{c_1 = \frac{1}{4}}$ $\boxed{c_2 = \frac{1}{4}}$

$\therefore$ Solution to BVP is $y = \frac{1}{4}\sqrt{x} + \frac{1}{4x} + \frac{x}{2}$

whenever degree of polynomial is same as degree of polynomial then it is called legendre eq.

Legendre Equation:

$$(ax+b)^n \frac{d^n y}{dx^n} + (ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + (ax+b) \frac{dy}{dx} + a_n y = r(x)$$

$[a=1,\ b=0] \rightarrow$ (Euler-Cauchy)

If $(ax+b) = e^t$

$(ax+b)D = a\theta,\quad (ax+b)^2 D^2 = a^2\theta(\theta-1),$

$(ax+b)^3 D^3 = a^3\theta(\theta-1)(\theta-2)$

where $D = \frac{d}{dx}$ & $\theta = \frac{d}{dt}$

Ques: $(3x+1)^2 y'' + (3x+1)y' + y = 6x$

Sol^n: $\underbrace{(3x+1)^2 \frac{d^2y}{dx^2} + (3x+1)\frac{dy}{dx}}$ $+ y = 6x.$

$\hookrightarrow$ [legendre's eq.]

$\downarrow$

NHLDEWVC

$(3x+1)^2 D^2 y + (3x+1)Dy + y = 6x$

$[(3x+1)^2 D^2 + (3x+1)D + 1]y = 6x.$

$3x+1 = e^t$

$(3x+1)D = 3\theta,\ (3x+1)^2 D^2 = 3^2\theta(\theta-1) = 9\theta(\theta-1)$

where $D = \frac{d}{dx},\ \theta = \frac{d}{dt}$

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$\Rightarrow [9\theta^2 - 9\theta + 3\theta + 1]y = 6\left[\frac{e^t - 1}{3}\right] = 2[e^t - 1]$

$\Rightarrow (9\theta^2 - 6\theta + 1)y = 2e^t - 1$ [NHLDEWC]

A.E, $9m^2 - 6m + 1 = 0 \Rightarrow (3m+1)^2 = 0$

$m = \frac{1}{3}, \frac{-1}{3}$

$y_c = (C_1 + C_2 t)e^{1/3 t}$

Now, $\frac{1}{9\theta^2 - 6\theta + 1}[2e^t - 2]$

$= \frac{1}{9\theta^2 - 6\theta + 1} 2[e^t] - \frac{1}{9\theta^2 + 6\theta + 1}(2)$

$= 2\frac{1}{[9-6+1]}e^t - 2\frac{1}{9\theta^2 - 6\theta + 1}e^{(0)t}$

$\therefore$ G.S is $y = y_c + y_p = (C_1 + C_2 t)e^{t/3} + \frac{e^t}{2} - 2$

$[C_1 + C_2 \log(3x+1)](3x+1)^{1/3} + \frac{3x+1}{2} - 2$